

# $L^p$ estimates for the maximal singular integral in terms of the singular integral

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## Abstract

This paper continues the study, initiated in the works [MOV] and [MOPV], of the problem of controlling the maximal singular integral  $T^*f$  by the singular integral  $Tf$ . Here  $T$  is a smooth homogeneous Calderón-Zygmund singular integral operator of convolution type. We consider two forms of control, namely, in the weighted  $L^p(\omega)$  norm and via pointwise estimates of  $T^*f$  by  $M(Tf)$  or  $M^2(Tf)$ , where  $M$  is the Hardy-Littlewood maximal operator and  $M^2 = M \circ M$  its iteration. The novelty with respect to the aforementioned works, lies in the fact that here  $p$  is different from 2 and the  $L^p$  space is weighted.

## 1 Introduction

Let  $T$  be a smooth homogeneous Calderón-Zygmund singular integral operator on  $\mathbb{R}^n$  with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n} \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (1)$$

where  $\Omega$  is a homogeneous function of degree 0 whose restriction to the unit sphere  $S^{n-1}$  is  $C^\infty$  and satisfies the cancellation property

$$\int_{|x|=1} \Omega(x) d\sigma(x) = 0,$$

$\sigma$  being the normalized surface measure in  $S^{n-1}$ . Thus,  $Tf$  is the principal value convolution operator

$$Tf(x) = \text{p.v.} \int f(x-y)K(y)dy \equiv \lim_{\varepsilon \rightarrow 0} T^\varepsilon f(x), \quad (2)$$

where  $T^\varepsilon f$  is the truncated operator at level  $\varepsilon$  defined by

$$T^\varepsilon f(x) = \int_{|x-y|>\varepsilon} f(x-y)K(y)dy.$$

For  $f \in L^p$ ,  $1 \leq p < \infty$ , the limit in (2) exists for almost all  $x$ . One says that the operator  $T$  is even (or odd) if the kernel (1) is even (or odd), that is, if  $\Omega(-x) = \Omega(x)$ ,  $x \in \mathbb{R}^n \setminus \{0\}$  (or  $\Omega(-x) = -\Omega(x)$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ ). Let  $T^*$  be the maximal singular integral

$$T^*f(x) = \sup_{\varepsilon > 0} |T^\varepsilon f(x)|, \quad x \in \mathbb{R}^n.$$

In this paper we consider the problem of characterizing those smooth Calderón-Zygmund operators for which one can control  $T^*f$  by  $Tf$  in the weighted  $L^p$  norm

$$\|T^*f\|_{L^p(\omega)} \leq C\|Tf\|_{L^p(\omega)}, \quad f \in L^p(\omega), \text{ and } \omega \in A_p, \quad (3)$$

where  $A_p$  is the Muckenhoupt class of weights (see below for the definition). A stronger way of saying that  $T^*$  is controlled by  $T$  is the pointwise inequality

$$T^*f(x) \leq C(M^s(Tf)(x)), \quad x \in \mathbb{R}^n, \quad s \in \{1, 2\}, \quad (4)$$

where  $M$  denotes the Hardy-Littlewood maximal operator and  $M^2 = M \circ M$  its iteration. For the case  $p = 2$  and  $\omega = 1$ , the relationship between (3) and (4) has been studied in [MOV] for even kernels and in [MOPV] for odd kernels (see also [MV]). We will prove that, for any  $1 < p < \infty$  and  $\omega \in A_p$ , the class of operators satisfying (3) coincides with the family of operators obtained for  $p = 2$  and  $\omega = 1$ , thus giving an affirmative answer to Question 1 of [MOV, p. 1480]. Our main result states that for smooth Calderón-Zygmund operators, inequality (4) (with  $s$  depending on the parity of the kernel) is equivalent to (3) and also is equivalent to an algebraic condition involving the expansion of  $\Omega$  in spherical harmonics.

Now we need to introduce some notation. The homogeneous function  $\Omega$ , like any square-integrable function in  $S^{n-1}$  with zero integral, has an expansion in spherical harmonics of the form

$$\Omega(x) = \sum_{j=1}^{\infty} P_j(x), \quad x \in S^{n-1}, \quad (5)$$

where  $P_j$  is a homogeneous harmonic polynomial of degree  $j$ . For the case of even operators in the above sum we only have the even terms  $P_{2j}$  and for the odd case we only have the polynomials of odd degree  $P_{2j+1}$ . In any case, when  $\Omega$  is infinitely differentiable on the unit sphere one has that, for each positive integer  $M$ ,

$$\sum_{j=1}^{\infty} j^M \|P_j\|_{\infty} < \infty, \quad (6)$$

where the supremum norm is taken on  $S^{n-1}$ . When  $\Omega$  is of the form

$$\Omega(x) = \frac{P(x)}{|x|^d}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

with  $P$  a homogeneous harmonic polynomial of degree  $d \geq 1$ , one says that  $T$  is a higher order Riesz transform. If the homogeneous polynomial  $P$  is not required to be harmonic, but has still zero integral on the unit sphere, then we call  $T$  a polynomial operator.

Let's recall the definition of Muckenhoupt weights. Let  $\omega$  be a non negative locally integrable function, and  $1 < p < \infty$ . Then  $\omega \in A_p$  if and only if there exists a constant  $C$  such that for all cubes  $Q \subset \mathbb{R}^n$

$$\left( \frac{1}{|Q|} \int_Q \omega \right) \left( \frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}} \right)^{p-1} \leq C.$$

The important fact worth noting is that Calderón-Zygmund operators and the Hardy-Littlewood maximal operator are bounded on  $L^p(\omega)$ , when  $1 < p < \infty$  and  $\omega$  belongs to  $A_p$ . See [Du, Chapter 7] or [Gr2, Chapter 9] to get more information on weights.

Now we state our result. We start with the case of even operators.

**Theorem 1.** *Let  $T$  be an even smooth homogeneous Calderón-Zygmund operator with kernel (1). Then the following are equivalent:*

(a)

$$T^*f(x) \leq CM(Tf)(x), \quad x \in \mathbb{R}^n.$$

(b) *If  $p \in (1, \infty)$  and  $\omega \in A_p$ , then*

$$\|T^*f\|_{L^p(\omega)} \leq C\|Tf\|_{L^p(\omega)}, \quad \text{for all } f \in L^p(\omega).$$

(c) *Assume that the expansion (5) of  $\Omega$  in spherical harmonics is*

$$\Omega(x) = \sum_{j=j_0}^{\infty} P_{2j}(x), \quad P_{2j_0} \neq 0.$$

*Then, for each  $j$  there exists a homogeneous polynomial  $Q_{2j-2j_0}$  of degree  $2j-2j_0$  such that  $P_{2j} = P_{2j_0}Q_{2j-2j_0}$  and  $\sum_{j=j_0}^{\infty} \gamma_{2j}Q_{2j-2j_0}(\xi) \neq 0$ ,  $\xi \in S^{n-1}$ . Here for a positive integer  $k$  we have set*

$$\gamma_k = i^{-k} \pi^{\frac{n}{2}} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{n+k}{2})}. \quad (7)$$

(d)

$$\|T^*f\|_{1,\infty} \leq C\|Tf\|_1, \quad \text{for all } f \in H^1(\mathbb{R}^n).$$

Recall that  $\|g\|_{1,\infty}$  denotes the weak  $L^1$  norm of  $g$  and  $H^1(\mathbb{R}^n)$  is the Hardy space. Calderón-Zygmund operators act on  $H^1$ . (For instance, see [Du, Chapter 6], [Gr2, Chapter 7] for more information on the Hardy space).

To get the above result for odd kernels we will replace the Hardy-Littlewood maximal operator in (a) by its iteration.

**Theorem 2.** *Let  $T$  be an odd smooth homogeneous Calderón-Zygmund operator with kernel (1). Then the following are equivalent:*

(a)

$$T^*f(x) \leq CM^2(Tf)(x), \quad x \in \mathbb{R}^n.$$

(b) *If  $p \in (1, \infty)$  and  $\omega \in A_p$  then*

$$\|T^*f\|_{L^p(\omega)} \leq C\|Tf\|_{L^p(\omega)}, \quad \text{for all } f \in L^p(\omega).$$

(c) *Assume that the expansion (5) of  $\Omega$  in spherical harmonics is*

$$\Omega(x) = \sum_{j=j_0}^{\infty} P_{2j+1}(x), \quad P_{2j_0+1} \neq 0.$$

*Then, for each  $j$  there exists a homogeneous polynomial  $Q_{2j-2j_0}$  of degree  $2j-2j_0$  such that  $P_{2j+1} = P_{2j_0+1}Q_{2j-2j_0}$  and  $\sum_{j=j_0}^{\infty} \gamma_{2j+1}Q_{2j-2j_0}(\xi) \neq 0$ ,  $\xi \in S^{n-1}$ , with  $\gamma_{2j+1}$  as in (7).*

Clearly, both in Theorem 1 as in Theorem 2, the condition (a) implies (b) is a consequence of the boundedness of the Hardy-Littlewood maximal operator on weighted  $L^p$  spaces. The proof of (c) implies (a) in Theorem 1 is proved in [MOV] and the same implication in Theorem 2 is proved in [MOPV]. So the only task to be done is to show that (b) implies (c) in both theorems (and (d)  $\Rightarrow$  (c) in Theorem 1). One of the crucial points in the proof of the implication (b)  $\Rightarrow$  (c) for the case  $p = 2$  and  $\omega = 1$  in [MOV] and [MOPV] is to use Plancherel Theorem to get a pointwise inequality to work with it. For  $p \neq 2$  we will get the corresponding pointwise inequality using properties of the Fourier transform of the kernels as  $L^p$  multipliers.

In Section 2 we introduce  $L^p$  Fourier multipliers and some tools to control their norm (see Lemma 1). Section 3 is devoted to the proof of (b)  $\Rightarrow$  (c), for polynomial operators. The general case is discussed in Section 4.

As usual, the letter  $C$  will denote a constant, which may be different at each occurrence and which is independent of the relevant variables under consideration.

## 2 Multipliers

Recall that, given  $1 \leq p < \infty$ , one denotes by  $\mathcal{M}_p(\mathbb{R}^n)$  the space of all bounded functions  $m$  on  $\mathbb{R}^n$  such that the operator

$$T_m(f) = (\hat{f} m)^\vee, \quad f \in \mathcal{S},$$

is bounded on  $L^p(\mathbb{R}^n)$  (or is initially defined in a dense subspace of  $L^p(\mathbb{R}^n)$  and has a bounded extension on the whole space). As usual,  $\mathcal{S}$  denotes the space of Schwartz

functions,  $\hat{f}$  is the Fourier transform of  $f$  and  $f^\vee$  the inverse Fourier transform. The norm of  $m$  in  $\mathcal{M}_p(\mathbb{R}^n)$  is defined as the norm of the bounded linear operator  $T_m : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ . Elements of the space  $\mathcal{M}_p(\mathbb{R}^n)$  are called  $L^p$  (Fourier) multipliers. Similarly, we speak of  $L^p(\omega)$  multipliers. It is well known that  $\mathcal{M}_2$ , the set of all  $L^2$  multipliers, is  $L^\infty$  and that  $\mathcal{M}_1(\mathbb{R}^n)$  is the set of Fourier transforms of finite Borel measures on  $\mathbb{R}^n$ . The basic theory on multipliers may be found for example in the monographs [Du],[Gr1].

Let  $0 \leq \phi \leq 1$  be a smooth function such that  $\phi(\xi) = 1$  if  $|\xi| \leq \frac{1}{2}$ , and  $\phi(\xi) = 0$  if  $|\xi| \geq 1$ . Given  $\xi_0 \in \mathbb{R}^n$ , we define  $\phi_\delta(\xi) = \phi(\frac{\xi - \xi_0}{\delta})$ . Consider  $m \in L^\infty$  such that  $m$  is continuous in some neighbourhood of  $\xi_0$  with  $m(\xi_0) = 0$ . It is clear, by Plancherel Theorem, that the norm of  $m\phi_\delta$  in  $\mathcal{M}_2$  approaches zero when  $\delta \rightarrow 0$ . We ask if the same result holds when  $m$  is an  $L^p$  multiplier. Adding some regularity to  $m$  we get a positive answer.

**Lemma 1.** *Let  $\xi_0 \in \mathbb{R}^n$ ,  $0 < \delta \leq \delta_0$  and  $m \in \mathcal{M}_p \cap \mathcal{C}^n(B(\xi_0, \delta_0))$  with  $m(\xi_0) = 0$ . Let  $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ ,  $0 \leq \phi \leq 1$  such that  $\phi(\xi) = 1$  if  $|\xi| \leq \frac{1}{2}$ , and  $\phi(\xi) = 0$  if  $|\xi| \geq 1$ . Set  $\phi_\delta(\xi) = \phi(\frac{\xi - \xi_0}{\delta})$  and let  $T_{m\phi_\delta}$  be the operator with multiplier  $m\phi_\delta$ .*

1. *If  $\omega \in A_p$ ,  $1 < p < \infty$ , then  $\|T_{m\phi_\delta}\|_{L^p(\omega) \rightarrow L^p(\omega)} \rightarrow 0$ , when  $\delta \rightarrow 0$ .*
2.  *$\|T_{m\phi_\delta}\|_{L^1 \rightarrow L^{1,\infty}} \rightarrow 0$ , when  $\delta \rightarrow 0$ .*
3.  *$\|T_{m\phi_\delta}\|_{H^1 \rightarrow L^1} \rightarrow 0$ , when  $\delta \rightarrow 0$ .*

To prove Lemma 1, we use the next theorem due to Kurtz and Wheeden. Following [KW], we say that a function  $m$  belongs to the class  $M(s, l)$  if

$$m_{s,l} := \sup_{R>0} \left( R^{s|\alpha|-n} \int_{R<|x|<2R} |D^\alpha m(x)|^s dx \right)^{1/s} < +\infty, \text{ for all } |\alpha| \leq l, \quad (8)$$

where  $s$  is a real number greater or equal to 1,  $l$  a positive integer and  $\alpha = (\alpha_1, \dots, \alpha_n)$  a multiindex of nonnegative integers.

**Theorem 3.** [KW, p. 344] *Let  $1 < s \leq 2$  and  $m \in M(s, n)$ .*

1. *If  $1 < p < \infty$  and  $\omega \in A_p$ , then there exists a constant  $C$ , independent of  $f$ , such that*

$$\|T_m f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

2. *There exists a constant  $C$ , independent of  $f$  and  $\lambda$ , such that*

$$|\{x \in \mathbb{R}^n : |T_m f(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L^1}, \quad \lambda > 0.$$

3. *There exists a constant  $C$ , independent of  $f$ , such that*

$$\|T_m f\|_{L^1} \leq C \|f\|_{H^1}.$$

Analyzing the proof we check that, in all cases, the constant  $C$ , which appears in the statements 1, 2 and 3 of the previous Theorem, depends linearly on the constant  $m_{s,n}$  defined at (8). We also remark that when  $\omega = 1$  the proof can be adapted to the case  $H^1 \rightarrow L^1$ , so we get statement 3 which is not explicitly written in [KW].

*Proof of Lemma 1.* Using Theorem 3 we only need to prove that the multiplier  $m\phi_\delta$  is in  $M(s, n)$  for some  $1 < s \leq 2$ , and the constant  $m_{s,n}$  tends to 0 if  $\delta$  tends to 0.

Assume that  $\xi_0 \neq 0$  and that  $\delta < \delta_0$  is small enough. For  $|\alpha| \leq n$ , using Leibniz rule one has

$$\begin{aligned} & \sup_{R>0} \left( R^{s|\alpha|-n} \int_{R<|\xi|<2R} |D^\alpha(m\phi_\delta)(\xi)|^s d\xi \right)^{1/s} \\ &= \sup_{R>0} \left( R^{s|\alpha|-n} \int_{\{R<|\xi|<2R\} \cap B(\xi_0, \delta)} |D^\alpha(m\phi_\delta)(\xi)|^s d\xi \right)^{1/s} \\ &\leq C|\xi_0|^{|\alpha|-\frac{n}{s}} \left( \int_{B(\xi_0, \delta)} |D^\alpha(m\phi_\delta)(\xi)|^s d\xi \right)^{1/s} \\ &\leq C|\xi_0|^{|\alpha|-\frac{n}{s}} \left( \sum_{\beta_i \leq \alpha_i, 1 \leq i \leq n} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n} \int_{B(\xi_0, \delta)} |D^{\alpha-\beta}(m)(\xi) D^\beta(\phi_\delta)(\xi)|^s d\xi \right)^{1/s}. \end{aligned}$$

Now we will get a bound for each term in the above sum. In order to get it, we consider different cases. In all the cases we will use that for any multiindex  $\alpha$  we have  $|D^\alpha \phi_\delta(\xi)| \lesssim \frac{1}{\delta^{|\alpha|}}$  and that the modulus of continuity of  $m$ , denoted by  $\omega(m, \xi_0, \delta)$ , satisfies  $\omega(m, \xi_0, \delta) \leq C\delta$ .

Case 1.  $|\alpha| = n$ .

For  $\beta = \alpha$  one has that

$$\begin{aligned} \int_{B(\xi_0, \delta)} |D^{\alpha-\beta}(m)(\xi) D^\beta(\phi_\delta)(\xi)|^s d\xi &= \int_{B(\xi_0, \delta)} |m(\xi)|^s |D^\alpha(\phi_\delta)(\xi)|^s d\xi \\ &\leq C \frac{1}{\delta^{ns}} |\omega(m, \xi_0, \delta)|^s \delta^n \\ &\leq C \delta^{s+n-ns} \end{aligned}$$

and this term tends to 0 as  $\delta$  tends to 0 taking  $1 < s < \frac{n}{n-1}$ . For the remaining terms, that is  $\alpha \neq \beta$ , we have

$$\begin{aligned} \int_{B(\xi_0, \delta)} |D^{\alpha-\beta}(m)(\xi) D^\beta(\phi_\delta)(\xi)|^s d\xi &\leq C \frac{1}{\delta^{|\beta|s}} \delta^n \\ &= C \delta^{n-s|\beta|} \\ &\leq C \delta^{s+n-ns}, \end{aligned}$$

where the derivatives of  $m$  are bounded by a constant, and the last inequality holds when  $\delta$  is small enough. So, if  $1 < s < \frac{n}{n-1}$ , this term goes to 0 as  $\delta$  goes to 0.

Case 2.  $|\alpha| = k < n$ .

For  $|\beta| = |\alpha|$ , using the boundedness of the modulus of continuity of  $m$  we have

$$\begin{aligned} \int_{B(\xi_0, \delta)} |D^{\alpha-\beta}(m)(\xi) D^\beta(\phi_\delta)(\xi)|^s d\xi &= \int_{B(\xi_0, \delta)} |m(\xi)|^s |D^\alpha(\phi_\delta)(\xi)|^s d\xi \\ &\leq C \frac{1}{\delta^{ks}} |\omega(m, \xi_0, \delta)|^s \delta^n \\ &= C \delta^{s+n-ks} \\ &\leq C \delta^{s+n-ns} \end{aligned}$$

and this term, again, goes to 0 as  $\delta$  goes to 0, whenever  $1 < s < \frac{n}{n-1}$ .

Finally, if  $|\beta| < |\alpha|$ , one gets the same bound

$$\begin{aligned} \int_{B(\xi_0, \delta)} |D^{\alpha-\beta}(m)(\xi) D^\beta(\phi_\delta)(\xi)|^s d\xi &\leq C \frac{1}{\delta^{|\beta|s}} \delta^n \\ &= C \delta^{n-s|\beta|} \\ &\leq C \delta^{s+n-ns}. \end{aligned}$$

When  $\xi_0 = 0$  one has

$$\begin{aligned} &\sup_{R>0} \left( R^{s|\alpha|-n} \int_{R<|\xi|<2R} |D^\alpha(m\phi_\delta)(\xi)|^s d\xi \right)^{1/s} \\ &= \sup_{\delta \geq R>0} \left( R^{s|\alpha|-n} \int_{R<|\xi|<2R} |D^\alpha(m\phi_\delta)(\xi)|^s d\xi \right)^{1/s}. \end{aligned}$$

Observe that for  $|\alpha| > 0$ ,  $D^\alpha \phi_\delta$  lives on  $\{\delta/2 \leq |\xi| \leq \delta\}$ . Then, similar calculations complete the proof.  $\square$

To prove the first case of Lemma 1 there is another argument due to J. Duoandikoetxea. We thank him for providing us the following lemma. In fact, it is only necessary to assume that the multiplier  $m$  is continuous.

**Lemma 2.** *Let  $\xi_0 \in \mathbb{R}^n$ ,  $0 < \delta \leq \delta_0$ ,  $1 < q < 2$  and  $m \in \mathcal{M}_q \cap \mathcal{C}(B(\xi_0, \delta_0))$  with  $m(\xi_0) = 0$ . Set  $\phi_\delta(\xi)$  as above and let  $T_{m\phi_\delta}$  be the operator with multiplier  $m\phi_\delta$ .*

(a) *For any  $p \in (q, 2)$  we have*

$$\|T_{m\phi_\delta}\|_{L^p \rightarrow L^p} \longrightarrow 0, \quad \text{when } \delta \rightarrow 0.$$

(b) *Let  $\omega \in A_p$  with  $p \in (q, 2)$  and let  $s > 1$  such that  $\omega^s \in A_p$ . If  $m$  is an  $L^p(\omega^s)$  multiplier, then*

$$\|T_{m\phi_\delta}\|_{L^p(\omega) \rightarrow L^p(\omega)} \longrightarrow 0, \quad \text{when } \delta \rightarrow 0.$$

**Remark 1.** Clearly, a similar result holds when  $2 < p < q$ .

*Proof.* We first observe that  $\|T_{m\phi_\delta}\|_{L^2 \rightarrow L^2} = \|m\phi_\delta\|_\infty = \varepsilon(\delta)$  and  $\varepsilon(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  since  $m$  is continuous in  $\xi_0$ . On the other hand,  $\|m\phi_\delta\|_{\mathcal{M}_q} \leq \|\phi_\delta^\vee\|_{L^1} \|m\|_{\mathcal{M}_q} = C\|m\|_{\mathcal{M}_q}$ , where  $C$  is a constant independent of  $\delta$ . That is, for all  $\delta > 0$

$$\|T_{m\phi_\delta}f\|_q \leq M\|f\|_q$$

Then, applying the Riesz-Thorin theorem (e.g. [Gr1, p. 34]), for any  $p \in (q, 2)$  ( $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$ ) we have

$$\|T_{m\phi_\delta}f\|_p \leq M^{1-\theta} \varepsilon(\delta)^\theta \|f\|_p = \varepsilon_1(\delta) \|f\|_p, \quad (9)$$

where  $\varepsilon_1(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  and (a) is proved. For proving (b), since  $\omega^s \in A_p$  and  $\phi_\delta$  is a cutoff smooth function, note that

$$\|T_{m\phi_\delta}f\|_{L^p(\omega^s)} \leq C\|f\|_{L^p(\omega^s)}, \quad (10)$$

where one can check that  $C$  is a constant independent of  $\delta$ . Finally, from (9) and (10), applying the interpolation theorem with change of measure of Stein-Weiss (e.g. [BeL, p. 115]), we get

$$\|T_{m\phi_\delta}f\|_{L^p(\omega)} \leq C^{1/s} \varepsilon_1(\delta)^{1-1/s} \|f\|_{L^p(\omega)}$$

as desired. □

### 3 The polynomial case

As we remarked in the Introduction, to have a complete proof of theorems 1 and 2 only remains to prove that (b) implies (c) (and (d) implies (c) in Theorem 1). Our procedure to get the above implications follows essentially the arguments used in [MOV] and [MOPV]. The main difficulty to overcome is that for  $p \neq 2$ , we cannot apply Plancherel Theorem and we replace it by a Fourier multiplier argument.

We begin with the proof of (b) implies (c) in Theorem 1 for the case  $\omega = 1$ . Then we show how to adapt this proof to the case with weights, to the case of odd operators and to the case of weak  $L^1$ . Thus, we assume that  $T$  is an even polynomial operator with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P_2(x)}{|x|^{2+n}} + \frac{P_4(x)}{|x|^{4+n}} + \dots + \frac{P_{2N}(x)}{|x|^{2N+n}}, \quad x \neq 0,$$

where  $P_{2j}$  is a homogeneous harmonic polynomial of degree  $2j$ . Each term has the multiplier (see [St, p. 73])

$$\left( \text{p.v.} \frac{P_{2j}(x)}{|x|^{2j+n}} \right)^\wedge (\xi) = \gamma_{2j} \frac{P_{2j}(\xi)}{|\xi|^{2j+n}},$$



Then,

$$\widehat{\text{p.v.}K}(\xi) = \frac{Q(\xi)}{|\xi|^{2N}}, \quad \xi \neq 0,$$

where  $Q$  is the homogeneous polynomial of degree  $2N$  defined by

$$Q(x) = \gamma_2 P_2(x) |x|^{2N-2} + \dots + \gamma_{2j} P_{2j}(x) |x|^{2N-2j} + \dots + \gamma_{2N} P_{2N}(x).$$

We want to obtain a convenient expression for the function  $K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}$ , the kernel  $K$  off the unit ball  $B$  (see (12)). To find it, we need a simple technical lemma which we state without proof.

**Lemma 3.** [MOV, p. 1435] *Assume that  $\varphi$  is a radial function of the form*

$$\varphi(x) = \varphi_1(|x|)\chi_B(x) + \varphi_2(|x|)\chi_{\mathbb{R}^n \setminus \overline{B}}(x),$$

where  $\varphi_1$  is continuously differentiable on  $[0, 1)$  and  $\varphi_2$  on  $(1, \infty)$ . Let  $L$  be a second order linear differential operator with constant coefficients. Then the distribution  $L\varphi$  satisfies

$$L\varphi = L\varphi(x)\chi_B(x) + L\varphi(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x),$$

provided  $\varphi_1, \varphi_1', \varphi_2$  and  $\varphi_2'$  extend continuously to the point 1 and the two conditions

$$\varphi_1(1) = \varphi_2(1), \quad \varphi_1'(1) = \varphi_2'(1)$$

are satisfied.

Consider the differential operator  $Q(\partial)$  defined by the polynomial  $Q(x)$  above and let  $E$  be the standard fundamental solution of the  $N$ -th power  $\Delta^N$  of the Laplacian. Then  $Q(\partial)E = \text{p.v.}K(x)$ , which may be verified by taking the Fourier transform of both sides. The concrete expression of  $E(x) = |x|^{2N-n}(a(n, N) + b(n, N) \log |x|^2)$  (e.g. [MOV, p. 1464]) is not important now, just note that it is a radial function. Consider the function

$$\varphi(x) = E(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) + (A_0 + A_1|x|^2 + \dots + A_{2N-1}|x|^{4N-2})\chi_B(x),$$

where  $B$  is the open ball of radius 1 centered at origin and the constants  $A_0, A_1, \dots, A_{2N-1}$  are chosen as follows. Since  $\varphi(x)$  is radial, the same is true for  $\Delta^j \varphi$  if  $j$  is a positive integer. Thus, in order to apply  $N$  times Lemma 3, one needs  $2N$  conditions, which (uniquely) determine  $A_0, A_1, \dots, A_{2N-1}$ . Therefore, for some constants  $\alpha_1, \alpha_2, \dots, \alpha_{N-1}$ ,

$$\Delta^N \varphi = (\alpha_0 + \alpha_1|x|^2 + \dots + \alpha_{N-1}|x|^{2(N-1)})\chi_B(x) = b(x), \quad (11)$$

where the last identity is the definition of  $b$ . Let's remark that  $b$  is a bounded function supported in the unit ball and it only depends on  $N$  and not on the kernel  $K$ . Since

$$\varphi = E * \Delta^N \varphi,$$

taking derivatives of both sides we obtain

$$Q(\partial)\varphi = Q(\partial)E * \Delta^N \varphi = \text{p.v.}K(x) * b = T(b).$$

On the other hand, applying Lemma 3,

$$Q(\partial)\varphi = K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) + Q(\partial)(A_0 + A_1|x|^2 + \dots + A_{2N-1}|x|^{4N-2})(x)\chi_B(x).$$

We write

$$S(x) := -Q(\partial)(A_0 + A_1|x|^2 + \dots + A_{2N-1}|x|^{4N-2})(x),$$

and we get

$$K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(b)(x) + S(x)\chi_B(x). \quad (12)$$

Let's remark that  $S$  will be null when  $Q$  is a harmonic polynomial (see [MOV, p. 1437]). Consequently

$$T^1 f = T(b) * f + S\chi_B * f.$$

Our assumption is the  $L^p$  estimate between  $T^*$  and  $T$ . Since the truncated operator  $T^1$  at level 1 is obviously dominated by  $T^*$ , we have

$$\begin{aligned} \|S\chi_B * f\|_p &\leq \|T^1 f\|_p + \|Tb * f\|_p \\ &\leq \|T^* f\|_p + \|b * Tf\|_p \\ &\leq C\|Tf\|_p + \|b\|_1 \|Tf\|_p \\ &= C\|Tf\|_p, \end{aligned} \quad (13)$$

that is, for any  $f \in L^p$

$$\|S\chi_B * f\|_p \leq C\|\text{p.v.}K * f\|_p. \quad (14)$$

If  $p = 2$ , we can use Plancherel and this  $L^2$  inequality translates into a pointwise inequality between the Fourier multipliers:

$$|\widehat{S\chi_B}(\xi)| \leq C|\widehat{\text{p.v.}K}(\xi)| = \frac{Q(\xi)}{|\xi|^{2N}}, \quad \xi \neq 0. \quad (15)$$

If  $p \neq 2$  we must resort to Fourier multipliers to get (15). We observe that the multipliers we are dealing with,  $\widehat{S\chi_B}$  and  $\widehat{\text{p.v.}K}$ , are in  $\mathcal{C}^\infty \setminus \{0\}$  and in  $\mathcal{M}_p$ . Let  $\xi_0 \neq 0$ , we write

$$\begin{aligned} \widehat{S\chi_B}(\xi) &= \widehat{S\chi_B}(\xi)(\xi_0) + E_1(\xi) \quad \text{with} \quad E_1(\xi) = \widehat{S\chi_B}(\xi) - \widehat{S\chi_B}(\xi_0) \\ \widehat{\text{p.v.}K}(\xi) &= \widehat{\text{p.v.}K}(\xi)(\xi_0) + E_2(\xi) \quad \text{with} \quad E_2(\xi) = \widehat{\text{p.v.}K}(\xi) - \widehat{\text{p.v.}K}(\xi_0) \end{aligned}$$

and so

$$\|\text{p.v.}K * f\|_p \leq |\widehat{\text{p.v.}K}(\xi_0)| \|f\|_p + \|T_{E_2} f\|_p, \quad (16)$$

$$\|S\chi_B * f\|_p \geq |\widehat{S\chi_B}(\xi_0)| \|f\|_p - \|T_{E_1} f\|_p, \quad (17)$$

where  $T_{E_i}$  denotes the operator with multiplier  $E_i$  ( $i = 1, 2$ ). Using (17), (14) and (16) consecutively, we get

$$\begin{aligned} |\widehat{S\chi_B}(\xi_0)| \|f\|_p - \|T_{E_1}f\|_p &\leq \|S\chi_B * f\|_p \\ &\leq C\|\text{p.v.}K * f\|_p \\ &\leq C(|\widehat{\text{p.v.}K}(\xi_0)| \|f\|_p + \|T_{E_2}f\|_p) \end{aligned}$$

and therefore

$$|\widehat{S\chi_B}(\xi_0)| \leq C \left( |\widehat{\text{p.v.}K}(\xi_0)| + \frac{\|T_{E_2}f\|_p}{\|f\|_p} + \frac{\|T_{E_1}f\|_p}{\|f\|_p} \right), \quad \xi_0 \neq 0. \quad (18)$$

Now, choosing appropriate functions in (18) we obtain the pointwise inequality. Let  $\phi_\delta(\xi) = \phi(\frac{\xi - \xi_0}{\delta})$  as in Lemma 1 and define  $g_\delta \in \mathcal{S}(\mathbb{R}^n)$  by  $\widehat{g_\delta}(\xi) = \phi_\delta(\xi)$ . Then  $T_{E_j}g_\delta = T_{E_j}(g_{2\delta} * g_\delta) = T_{E_j\phi_{2\delta}}(g_\delta)$ , because  $\phi_{2\delta} = 1$  on the support of  $\phi_\delta$ . Changing  $f$  by  $g_\delta$  in (18) we have

$$\begin{aligned} |\widehat{S\chi_B}(\xi_0)| &\leq C \left( |\widehat{\text{p.v.}K}(\xi_0)| + \frac{\|T_{E_2\phi_{2\delta}}g_\delta\|_p}{\|g_\delta\|_p} + \frac{\|T_{E_1\phi_{2\delta}}g_\delta\|_p}{\|g_\delta\|_p} \right) \\ &\leq C \left( |\widehat{\text{p.v.}K}(\xi_0)| + \|T_{E_2\phi_{2\delta}}\|_{L^p \rightarrow L^p} + \|T_{E_1\phi_{2\delta}}\|_{L^p \rightarrow L^p} \right). \end{aligned}$$

Applying Lemma 1 to the multipliers  $E_j$  we prove that the two last terms tend to zero as  $\delta$  tends to zero. So, for  $\omega = 1$ , we get (15) and from here we would follow the arguments in [MOV, p. 1457].

For the weighted case we must be careful with the inequalities in (13). In general, the inequality  $\|f * F\|_{L^p(\omega)} \leq C\|f\|_1\|F\|_{L^p(\omega)}$  is not satisfied. That is, we can not control  $\|b * Tf\|_{L^p(\omega)}$  by a constant times  $\|b\|_1\|Tf\|_{L^p(\omega)}$ . However, in the even case  $b$  is a bounded function supported in the unit ball and so

$$|(b * Tf)(x)| = \left| \int_{|x-y|<1} b(x-y)Tf(y) dy \right| \leq CM(Tf)(x).$$

Moreover

$$\|b * Tf\|_{L^p(\omega)} \leq C\|Tf\|_{L^p(\omega)},$$

because  $\omega \in A_p$ . So,  $\|S\chi_B * f\|_{L^p(\omega)} \leq C\|\text{p.v.}K * f\|_{L^p(\omega)}$  and proceeding as above, we would get (15).

The proof of (b) implies (c) in Theorem 2 can be handled in much the same way. The only significant difference, because now the polynomial is odd, lies on the function  $b$  in (12), which is not supported in the unit ball but it is a BMO function satisfying the decay  $|b(x)| \leq C|x|^{-n-1}$  if  $|x| > 2$  (see [MOPV, section 4]). In any case,  $b \in L^1$  and the set of inequalities (13) remains valid for the case  $\omega = 1$ .

On the other hand, for any  $\omega$  in the Muckenhoupt class we write, arguing as in [MOPV, p. 3675],

$$\begin{aligned} |(b * Tf)(x)| &= \left| \int_{|x-y|<2} (b(x-y) - b_{B(0,2)}) Tf(y) dy \right| + \\ &\quad + |b_{B(0,2)}| \int_{|x-y|<2} |Tf(y)| dy + \int_{|x-y|>2} |b(x-y)| |Tf(y)| dy \\ &= I + II + III, \end{aligned}$$

where  $b_{B(0,2)} = |B(0,2)|^{-1} \int_{B(0,2)} b$ . To estimate the local term  $I$  we use the generalized Hölder's inequality and the pointwise equivalence  $M_{L(\log L)} f(x) \simeq M^2 f(x)$  ([P]) to get

$$|I| \leq C \|b\|_{BMO} \|Tf\|_{L(\log L), B(x,2)} \leq CM^2(Tf)(x).$$

Notice that  $b_{B(0,2)}$  is a dimensional constant. Hence

$$|II| \leq CM(Tf)(x).$$

Finally, from the decay of  $b$  we obtain

$$|III| \leq C \int_{|x-y|>2} \frac{|Tf(y)|}{|x-y|^{n+1}} dy \leq CM(Tf)(x),$$

by using a standard argument which consists in estimating the integral on the annuli  $\{2^k \leq |x-y| < 2^{k+1}\}$ . Therefore

$$|(b * Tf)(x)| \leq CM^2(Tf)(x). \quad (19)$$

So, we obtain

$$\|b * Tf\|_{L^p(\omega)} \leq C \|Tf\|_{L^p(\omega)},$$

because  $\omega \in A_p$ . Then,  $\|S\chi_B * f\|_{L^p(\omega)} \leq C \|\text{p.v.} K * f\|_{L^p(\omega)}$  and we get (15).

It remains to prove that (d) implies (c) in Theorem 1. To get this implication we need to precise some properties of the functions  $g_\delta$  that we explain below. First of all, note that  $g_\delta(x) = e^{ix\xi_0} \delta^n g(\delta x)$  where  $\hat{g} = \phi$ . So it is clear that the norms  $\|g_\delta\|_1 = \|g\|_1$  and  $\|g_\delta\|_{1,\infty} = \|g\|_{1,\infty}$  do not depend on the parameter  $\delta > 0$ . When  $\delta < |\xi_0|$ , since  $\int g_\delta(x) dx = \phi_\delta(0) = 0$  and  $g_\delta \in \mathcal{S}(\mathbb{R}^n)$ , we have that  $g_\delta \in H^1$ . But, some computations are required to check that  $\|g_\delta\|_{H^1} \leq C$  with constant  $C$  independent of  $\delta$ .

**Lemma 4.** *When  $0 < \delta < |\xi_0|$ ,  $\|g_\delta\|_{H^1} \leq C$  with constant  $C$  independent of  $\delta$ .*

*Proof.* We have  $g_\delta(x) = e^{ix\xi_0} \delta^n g(\delta x)$  with  $g \in \mathcal{S}(\mathbb{R}^n)$  and  $\int g_\delta = 0$ . Set  $F_0^\delta(x) = \chi_{B(0,\delta^{-1})}(x)$  and, for  $j \geq 1$ ,  $F_j^\delta(x) = \chi_{B(0,2^j\delta^{-1})}(x) - \chi_{B(0,2^{j-1}\delta^{-1})}(x)$ . Note that

$\sum_{j=0}^{\infty} F_j^\delta(x) \equiv 1$ . Consider the atomic decomposition of  $g_\delta$

$$\begin{aligned} g_\delta(x) &= \sum_{j=0}^{\infty} (g_\delta(x) - c_j^\delta) F_j^\delta(x) + \sum_{j=0}^{\infty} [(c_j^\delta + d_j^\delta) F_j^\delta(x) - d_{j+1}^\delta F_{j+1}^\delta(x)] \\ &:= \sum_{j=0}^{\infty} a_j^\delta(x) + \sum_{j=0}^{\infty} A_j^\delta(x), \end{aligned}$$

where  $c_j^\delta = \frac{\int g_\delta F_j^\delta}{\int F_j^\delta}$ ,  $d_0^\delta = 0$  and  $d_{j+1}^\delta = \frac{\int g_\delta (F_0^\delta + \dots + F_j^\delta)}{\int F_{j+1}^\delta}$ , so that  $\int a_j^\delta(x) dx = \int A_j^\delta(x) dx = 0$ . Note that  $a_j^\delta$  is supported in the ball  $B(0, 2^j \delta^{-1})$  and  $A_j^\delta$  is supported in  $B(0, 2^{j+1} \delta^{-1})$ .

Since  $g \in \mathcal{S}(\mathbb{R}^n)$  we have  $(1 + |z|^{n+1})|g(z)| \leq C$ . Then

$$|g_\delta(x) F_j^\delta(x)| = \delta^n |g(\delta x)| F_j^\delta(x) \leq \delta^n \sup_{|z| \sim 2^j} |g(z)| \leq C \left( \frac{\delta}{2^j} \right)^n 2^{-j} = \frac{C 2^{-j}}{|B(0, 2^j \delta^{-1})|}$$

and therefore

$$|c_j^\delta| = \left| \frac{\int g_\delta F_j^\delta}{\int F_j^\delta} \right| \leq \frac{C 2^{-j}}{|B(0, 2^j \delta^{-1})|}.$$

On the other hand,  $\int g_\delta (F_0^\delta + \dots + F_j^\delta) = \int_{|x| \geq 2^j \delta^{-1}} g_\delta(x) dx$ , because  $\int g_\delta = 0$ , and so

$$d_{j+1}^\delta = \frac{\int_{|x| \geq 2^j \delta^{-1}} g_\delta(x) dx}{\int F_{j+1}^\delta} \leq \frac{\int_{|z| \geq 2^j} |g(z)| dz}{|B(0, 2^{j+1} \delta^{-1})|} \leq \frac{C 2^{-j}}{|B(0, 2^{j+1} \delta^{-1})|}.$$

Consequently

$$\|a_j^\delta\|_{H^1} \leq \frac{C}{2^j} \quad \text{and} \quad \|A_j^\delta\|_{H^1} \leq \frac{C}{2^j}.$$

Therefore, for all  $\delta \in (0, |\xi_0|)$ ,  $\|g_\delta\|_{H^1} \leq C$  as we claimed. □

Finally, for functions  $f$  in  $H^1$ , and again using (12), we have

$$\begin{aligned} \|S_{\chi_B} * f\|_{1,\infty} &\leq 2(\|T^1 f\|_{1,\infty} + \|Tb * f\|_{1,\infty}) \\ &\leq C(\|T^* f\|_{1,\infty} + \|b * Tf\|_1) \\ &\leq C\|Tf\|_1 + \|b\|_1 \|Tf\|_1 \\ &= C\|Tf\|_1 = C\|\text{p.v.}K * f\|_1. \end{aligned}$$

Taking  $\xi_0 \neq 0$  and using the same notation as before, we have

$$\begin{aligned} \|\text{p.v.}K * f\|_1 &\leq |\widehat{\text{p.v.}K}(\xi_0)| \|f\|_1 + \|T_{E_2} f\|_1, \\ \|S_{\chi_B} * f\|_{1,\infty} &\geq \frac{1}{2} |\widehat{S_{\chi_B}}(\xi_0)| \|f\|_{1,\infty} - \|T_{E_1} f\|_{1,\infty} \end{aligned}$$

and consequently

$$|\widehat{S\chi_B}(\xi_0)| \leq C \left( |\widehat{\text{p.v.}K}(\xi_0)| \frac{\|f\|_1}{\|f\|_{1,\infty}} + \frac{\|T_{E_2}f\|_1}{\|f\|_{1,\infty}} + \frac{\|T_{E_1}f\|_{1,\infty}}{\|f\|_{1,\infty}} \right), \quad \xi_0 \neq 0.$$

Replacing  $f$  by  $g_\delta$  and using the properties of  $g_\delta$  (that is,  $\|g_\delta\|_1 = \|g\|_1$ ,  $\|g_\delta\|_{1,\infty} = \|g\|_{1,\infty}$  and Lemma 4) we obtain

$$\begin{aligned} |\widehat{S\chi_B}(\xi_0)| &\leq C \left( |\widehat{\text{p.v.}K}(\xi_0)| \frac{\|g_\delta\|_1}{\|g_\delta\|_{1,\infty}} + \frac{\|T_{E_2\phi_{2\delta}}g_\delta\|_1}{\|g_\delta\|_{1,\infty}} + \frac{\|T_{E_1\phi_{2\delta}}g_\delta\|_{1,\infty}}{\|g_\delta\|_{1,\infty}} \right) \\ &\leq C \left( |\widehat{\text{p.v.}K}(\xi_0)| \frac{\|g\|_1}{\|g\|_{1,\infty}} + \frac{\|T_{E_2\phi_{2\delta}}\|_{H^1 \rightarrow L^1} \|g_\delta\|_{H^1}}{\|g_\delta\|_{1,\infty}} + \frac{\|T_{E_1\phi_{2\delta}}\|_{L^1 \rightarrow L^{1,\infty}} \|g_\delta\|_1}{\|g_\delta\|_{1,\infty}} \right) \\ &\leq C \left( |\widehat{\text{p.v.}K}(\xi_0)| + \|T_{E_2\phi_{2\delta}}\|_{H^1 \rightarrow L^1} + \|T_{E_1\phi_{2\delta}}\|_{L^1 \rightarrow L^{1,\infty}} \right) \end{aligned}$$

and therefore, applying Lemma 1 on the right hand side of this inequality, we get

$$|\widehat{S\chi_B}(\xi_0)| \leq C |\widehat{\text{p.v.}K}(\xi_0)| \quad \xi_0 \neq 0$$

as desired.

## 4 The general case

In our procedure for the polynomial case, the function  $b$  has been crucial. It provides a convenient way to express the function  $K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}$ , where  $K$  is the kernel of the operator  $T$ . As we mentioned before,  $b$  only depends on the degree of the homogeneous polynomial and on the space  $\mathbb{R}^n$ . In the even case  $2N$  (see (11)),  $b = b_{2N}$  is the restriction to the unit ball of some polynomial of degree  $2N - 2$ . In the odd case  $2N + 1$ ,  $b_{2N+1}$  is a BMO function with certain decay at infinity. Until now, we did not need to pay attention to the size of the parameters appearing in the definition of  $b$  because the degree of the polynomial (either  $2N$  or  $2N + 1$ ) was fixed. In this section we require a control of the  $L^1$ ,  $L^\infty$  or BMO norms of  $b$ , as well as its decay at infinity. We summarize all we need in next lemma.

**Lemma 5.** *There exists a constant  $C$  depending only on  $n$  such that*

- (i)  $|\widehat{b_{2N}}(\xi)| \leq C$  and  $|\widehat{b_{2N+1}}(\xi)| \leq C$ ,  $\xi \in \mathbb{R}^n$ .
- (ii)  $\|b_{2N}\|_{L^\infty(B)} \leq C(2N)^{2n+2}$  and  $\|\nabla b_{2N}\|_{L^\infty(B)} \leq C(2N)^{2n+4}$ .
- (iii)  $\|b_{2N+1}\|_{\text{BMO}} \leq C(2N+1)^{2n}$  and  $\|b_{2N+1}\|_{L^2} \leq C(2N+1)^{2n}$ .
- (iv) If  $|x| > 2$  then  $|b_{2N+1}(x)| \leq C(2N+1)^{2n}|x|^{-n-1}$ .

*Proof.* Parts (i), (ii) and (iii) are proved in [MOV, Lemma 8] and [MOPV, Lemma 5]. It only remains to prove (iv).

Recall that  $\sigma$  denotes the normalized surface measure in  $S^{n-1}$ , and let  $h_1, \dots, h_d$  be an orthonormal basis of the subspace of  $L^2(d\sigma)$  consisting of all homogeneous harmonic polynomials of degree  $2N+1$ . As it is well known,  $d \simeq (2N+1)^{n-2}$ . As in the proof of Lemma 6 in [MOV] we have  $h_1^2 + \dots + h_d^2 = d$ , on  $S^{n-1}$ . Set

$$H_j(x) = \frac{1}{\gamma_{2N+1} \sqrt{d}} h_j(x), \quad x \in \mathbb{R}^n,$$

and let  $S_j$  be the higher order Riesz transform with kernel  $K_j(x) = H_j(x)/|x|^{2N+1+n}$ . The Fourier multiplier of  $S_j^2$  is

$$\frac{1}{d} \frac{h_j(\xi)^2}{|\xi|^{4N+2}}, \quad 0 \neq \xi \in \mathbb{R}^n,$$

and thus

$$\sum_{j=1}^d S_j^2 = \text{Identity}. \quad (20)$$

We use again (12), but now the second term at the right hand side vanishes because each  $h_j$  is harmonic (see [MOV], p. 1437). We get

$$K_j(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) = S_j(b_{2N+1})(x), \quad x \in \mathbb{R}^n, \quad 1 \leq j \leq d,$$

and so by (20)

$$b_{2N+1} = \sum_{j=1}^d S_j \left( K_j(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) \right). \quad (21)$$

Therefore we set

$$\begin{aligned} \sum_{j=1}^d S_j \left( K_j(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) \right) &= \sum_{j=1}^d S_j * S_j - \sum_{j=1}^d S_j (K_j(x) \chi_B(x)) \\ &= \delta_0 - \sum_{j=1}^d S_j (K_j(x) \chi_B(x)), \end{aligned}$$

where  $\delta_0$  is the Dirac delta at the origin. If  $|x| > 2$ , then

$$\begin{aligned} S_j(K_j(y) \chi_B(y))(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |y| < 1} K_j(x-y) K_j(y) dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |y| < 1} (K_j(x-y) - K_j(x)) K_j(y) dy. \end{aligned}$$

In this situation,

$$|K_j(x - y) - K_j(x)| \leq C \frac{|y|}{|x|^{n+1}} (\|H_j\|_\infty (2N + 1) + \|\nabla H_j\|_\infty),$$

hence

$$|S_j(K_j(y) \chi_B(y))(x)| \leq C \frac{\|H_j\|_\infty (2N + 1) + \|\nabla H_j\|_\infty}{|x|^{n+1}} \int_{|y| < 1} \frac{\|H_j\|_\infty}{|y|^{n-1}} dy$$

where the supremum norms are taken on  $S^{n-1}$ . Clearly

$$\|H_j\|_\infty = \frac{1}{\gamma_{2N+1}} \left\| \frac{h_j}{\sqrt{d}} \right\|_\infty \leq \frac{1}{\gamma_{2N+1}} \simeq (2N + 1)^{n/2}.$$

For the estimate of the gradient of  $H_j$  we use the inequality [St, p. 276]

$$\|\nabla H_j\|_\infty \leq C (2N + 1)^{n/2+1} \|H_j\|_2,$$

where the  $L^2$  norm is taken with respect to  $d\sigma$ . Since the  $h_j$  are an orthonormal system,

$$\|H_j\|_2 = \frac{1}{\sqrt{d} \gamma_{2N+1}} \simeq \frac{(2N + 1)^{n/2}}{(2N + 1)^{(n-2)/2}} \simeq 2N + 1.$$

Gathering the above inequalities we get, when  $|x| > 2$ ,

$$|S_j(K_j(y) \chi_B(y))(x)| \leq C \frac{(2N + 1)^{n+2}}{|x|^{n+1}}$$

and finally

$$|b_{2N+1}(x)| \leq Cd \frac{(2N + 1)^{n+2}}{|x|^{n+1}} \leq C \frac{(2N + 1)^{2n}}{|x|^{n+1}},$$

as claimed. □

Now, the kernel of the operator  $Tf = \text{p.v.} K * f$  is of the type  $K(x) = \frac{\Omega(x)}{|x|^n}$  being  $\Omega$  a  $C^\infty(S^{n-1})$  homogeneous function of degree 0, with vanishing integral on the sphere. Then,  $\Omega(x) = \sum_{j \geq 1}^\infty \frac{P_{2j}(x)}{|x|^{2j}}$  with  $P_{2j}$  homogeneous harmonic polynomials of degree  $2j$  when  $T$  is an even operator, and  $\Omega(x) = \sum_{j \geq 0}^\infty \frac{P_{2j+1}(x)}{|x|^{2j+1}}$  with  $P_{2j+1}$  homogeneous harmonic polynomials of degree  $2j + 1$  when  $T$  is an odd operator. The strategy consists in passing to the polynomial case by looking at a partial sum of the series above. Set, for each  $N \geq 1$ ,  $K_N(x) = \frac{\Omega_N(x)}{|x|^n}$ , where  $\Omega_N(x) = \sum_{j=1}^N \frac{P_{2j}(x)}{|x|^{2j}}$  (or  $\Omega_N(x) = \sum_{j=0}^N \frac{P_{2j+1}(x)}{|x|^{2j+1}}$  in the odd case), and let  $T_N$  be the operator with kernel  $K_N$ .



We begin by considering (b) implies (c) in Theorem 1 when  $\omega = 1$ , that is,  $T$  is even and our hypothesis is  $\|T^*f\|_p \leq C\|Tf\|_p$ ,  $f \in L^p(\mathbb{R}^n)$ . In this setting, the difficulty is that there is no obvious way of obtaining the inequality

$$\|T_N^*f\|_p \leq C\|T_Nf\|_p, \quad f \in L^p(\mathbb{R}^n). \quad (22)$$

Instead, we try to get (22) with  $\|T_Nf\|_p$  replaced by  $\|Tf\|_p$  in the right hand side plus an additional term which becomes small as  $N$  tends to  $\infty$ . We start by writing

$$\begin{aligned} \|T_N^1f\|_p &\leq \|T^1f\|_p + \|T^1f - T_N^1f\|_p \\ &\leq C\|Tf\|_p + \left\| \sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c} * f \right\|_p. \end{aligned} \quad (23)$$

By (12), and since every  $P_{2j}$  is harmonic, there exists a bounded function  $b_{2j}$  supported on  $B$  such that

$$\frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c}(x) = \text{p.v.} \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j}.$$

By Lemma 5 (ii), we have that  $\|b_{2j}\|_{L^1} \leq C\|b_{2j}\|_{L^\infty(B)} \leq C(2j)^{2n+2}$ , and thus

$$\begin{aligned} \left\| \sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c} * f \right\|_p &= \left\| \sum_{j>N} \text{p.v.} \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j} * f \right\|_p \\ &\leq \sum_{j>N} \left\| \frac{P_{2j}(x)}{|x|^{2j+n}} \right\|_{L^p \rightarrow L^p} \|b_{2j} * f\|_p \\ &\leq \sum_{j>N} \left\| \frac{P_{2j}(x)}{|x|^{2j+n}} \right\|_{L^p \rightarrow L^p} \|b_{2j}\|_1 \|f\|_p \\ &\leq C\|f\|_p \sum_{j>N} \left\| \frac{P_{2j}(x)}{|x|^{2j+n}} \right\|_{L^p \rightarrow L^p} (2j)^{2n+2} \\ &\leq C\|f\|_p \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty) (2j)^{2n+2}. \end{aligned} \quad (24)$$

The last inequality follows from a well-known estimate for Calderón-Zygmund operators (e.g. [Gr1, Theorem 4.3.3]). On the other hand,

$$K_N(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T_N(b_{2N})(x) + S_N(x) \chi_B(x)$$

and then

$$T_N^1f = \text{p.v.} K_N * b_{2N} * f + S_N \chi_B * f.$$

So, for each  $f \in L^p(\mathbb{R}^n)$ , using (23) and (24), we have the  $L^p$  inequality

$$\begin{aligned} \|S_N \chi_B * f\|_p &\leq \|T_N^1f\|_p + \|\text{p.v.} K_N * b_{2N} * f\|_p \\ &\leq C \left( \|Tf\|_p + \|f\|_p \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty) (2j)^{2n+2} + \|\text{p.v.} K_N * b_{2N} * f\|_p \right). \end{aligned}$$

We emphasize that the corresponding multipliers  $\widehat{S_N \chi_B}$ ,  $\widehat{\text{p.v.} K}$  and  $\widehat{\text{p.v.} K_N * b_{2N}} = \widehat{\text{p.v.} K_N b_{2N}}$  are in  $\mathcal{C}^\infty \setminus \{0\}$  and in  $\mathcal{M}_p$ . Therefore, proceeding as in the polynomial case, and applying Lemma 1 we obtain the pointwise estimate for  $\xi \neq 0$

$$\begin{aligned} |\widehat{S_N \chi_B}(\xi)| &\leq C \left( |\widehat{\text{p.v.} K}(\xi)| + |(\widehat{\text{p.v.} K_N} \cdot \widehat{b_{2N}})(\xi)| + \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty) (2j)^{2n+2} \right) \\ &\leq C \left( |\widehat{\text{p.v.} K}(\xi)| + |\widehat{\text{p.v.} K_N}(\xi)| + \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty) (2j)^{2n+2} \right), \end{aligned}$$

where in the last step we have used Lemma 5 (i), that is,  $|\widehat{b_{2N}}(\xi)| \leq C$ , for  $\xi \in \mathbb{R}^n$ .

The idea is now to take limits, as  $N$  goes to  $\infty$ , in the preceding inequality. By the definition of  $K_N$  and (6), the term on the right-hand side converges to  $C|\widehat{\text{p.v.} K}(\xi)|$ . The next task is to clarify how the left-hand side converges, but at this point we proceed as in [MOV, p. 1463] and we get the desired result.

This argument, which has been explained for the even case and  $\omega = 1$ , is also valid for the other cases, after taking into account the particular details listed below.

To get (b) implies (c) in Theorem 1 for any  $\omega \in A_p$ , we would use

$$\|b_{2j} * f\|_{L^p(\omega)} \leq C \|b_{2j}\|_{L^\infty(B)} \|Mf\|_{L^p(\omega)} \leq C (2j)^{2n+2} \|f\|_{L^p(\omega)}$$

to obtain the inequality analogous to (24)

In order to obtain (d) implies (c) in Theorem 1, note that if  $c_j > 0$  and  $\sum_{j=1}^\infty c_j = 1$ , then  $\|\sum g_j\|_{1,\infty} \leq \sum c_j^{-1} \|g_j\|_{1,\infty}$ . We have

$$\begin{aligned} \left\| \sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c} * f \right\|_{1,\infty} &= \left\| \sum_{j>N} \text{p.v.} \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j} * f \right\|_{1,\infty} \\ &\leq \sum_{j>N} j^2 \left\| \text{p.v.} \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j} * f \right\|_{1,\infty} \\ &\leq \sum_{j>N} j^2 \left\| \frac{P_{2j}(x)}{|x|^{2j+n}} \right\|_{L^1 \rightarrow L^{1,\infty}} \|b_{2j} * f\|_1 \\ &\leq \sum_{j>N} j^2 \left\| \frac{P_{2j}(x)}{|x|^{2j+n}} \right\|_{L^1 \rightarrow L^{1,\infty}} \|b_{2j}\|_1 \|f\|_1 \\ &\leq C \|f\|_1 \sum_{j>N} j^2 \left\| \frac{P_{2j}(x)}{|x|^{2j+n}} \right\|_{L^1 \rightarrow L^{1,\infty}} (2j)^{2n+2} \\ &\leq C \|f\|_1 \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty) (2j)^{2n+4}, \end{aligned}$$

and therefore, for all functions  $f \in H^1(\mathbb{R}^n)$ ,

$$\begin{aligned}
\|S_N \chi_B * f\|_{1,\infty} &\leq 2(\|T_N^1 f\|_{1,\infty} + \|\text{p.v.} K_N * b_{2N} * f\|_{1,\infty}) \\
&\leq 4(\|T^1 f\|_{1,\infty} + \|\sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c} * f\|_{1,\infty}) + 2\|\text{p.v.} K_N * b_{2N} * f\|_{1,\infty}) \\
&\leq C(\|Tf\|_1 + \|f\|_1 \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty) (2j)^{2n+4} + \\
&\quad + \|\text{p.v.} K_N * b_{2N} * f\|_{1,\infty}).
\end{aligned}$$

Again, using Lemma 1, Lemma 4 and Lemma 5, we obtain, for  $\xi \neq 0$ ,

$$|\widehat{S_N \chi_B}(\xi)| \leq C \left( |\widehat{\text{p.v.} K}(\xi)| + |\widehat{\text{p.v.} K_N}(\xi)| + \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty) (2j)^{2n+4} \right)$$

as desired.

The implication (b)  $\Rightarrow$  (c) in Theorem 2 can be adapted as follows.  $T$  is odd and the functions  $b_{2j+1}$  are in BMO. By Lemma 5, we have  $\|\widehat{b_{2j+1}}\|_\infty \leq C$ ,  $\|b_{2j+1}\|_{\text{BMO}} \leq C(2j+1)^{2n}$  and  $\|b_{2j+1}\|_2 \leq C(2j+1)^{2n}$ . Moreover,  $|b_{2j+1}(x)| \leq C(2j+1)^{2n}|x|^{-n-1}$  if  $|x| > 2$ . Then, proceeding in the same way as in the proof of (19), we get

$$\|b_{2j+1} * f\|_{L^p(\omega)} \leq C(2j+1)^{2n} \|f\|_{L^p(\omega)}$$

and so, the inequality analogous to (24) follows.

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